Phys 410
Spring 2013
Lecture #12 Summary
18 February, 2013

We considered un-driven damped oscillations produced by a damping force that is linear in velocity $m\ddot{x} + b\dot{x} + kx = 0$. This mechanical oscillator is a direct analog of the electrical oscillator made up of an inductor (L), resistor (R) and capacitor (C) in series. The charge on the capacitor plate $q(t)$ obeys the same equation: $L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0$. The analogy is strong, as shown in the following table.

Mechanical Oscillator	**Electrical Oscillator**
Position *x*	Charge on capacitor plate *q*
Mass *m*	Inductance *L*
Damping *b*	Resistance *R*
Spring constant *k*	Inverse Capacitance 1/*C*
Natural frequency $\omega_0 = [k/m]^{1/2}$	Natural frequency $\omega_0 = 1/[LC]^{1/2}$

Divide the mechanical equation through by mass m and define two important rates: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where $2\beta \equiv b/m$, and $\omega_0^2 \equiv k/m$. We tried a solution of the form $x(t) = e^{rt}$, and found an auxiliary equation with solution $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. The general solution is $x(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\ t}\right]$. The form of the solution depends critically on the relative size of the two rates β and ω_0.

1) Un-damped oscillator $\beta = 0$. The radical becomes $\sqrt{-\omega_0^2} = i\sqrt{\omega_0^2} = i\omega_0$, and the solution reverts to our previous results $x(t) = C_1 e^{i\omega_0\ t} + C_2 e^{-i\omega_0\ t}$.
2) Weak damping ($\beta < \omega_0$). The radical also produces a factor of "*i*", resulting in $x(t) = e^{-\beta t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$, with $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ a frequency lower than the un-damped natural frequency. This equation describes oscillatory motion under an exponentially damped envelope. The damping rate is β. One can re-write the solution as $x(t) = A\,e^{-\beta t}\cos(\omega_1 t - \delta)$.
3) Strong damping ($\beta > \omega_0$). In this case $\sqrt{\beta^2 - \omega_0^2}$ is real and the solution is $x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)\ t} + C_2 e^{-\left(\beta + \sqrt{\beta^2-\omega_0^2}\right)\ t}$. This is a sum of two negative exponentials, one of which decays faster than the other – there is no oscillation.

We next considered a *driven* damped harmonic oscillator. We take the driving function to harmonic in time at a new frequency called simply ω, which is an independent quantity from the natural frequency of the un-damped oscillator, called ω_0. The equation of motion is now $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$. We now employ a trick similar to that used to solve for the velocity of a charged particle in a uniform magnetic field. Consider the complementary problem of the same damped oscillator being driven by a force 90° out of phase, with solution $y(t)$: $\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$. Now define a complex combination of the two unknown functions $z(t) = x(t) + iy(t)$. Combine the two equations in the form of "x-equation" + i "y-equation". This can be written more succinctly as $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. Note that the solution to the original problem can be found from $x(t) = Re[z(t)]$.